Teacher Perspectives on Derivative

Alan Gil delos Santos
The University of Auckland
<santos@math.auckland.ac.nz>

Michael Thomas
The University of Auckland
<m.thomas@math.auckland.ac.nz>

The role of the teacher in the learning of mathematics is one which continues to be refined. Evidence continues to mount for the relationship between teachers' content knowledge and pedagogical content knowledge, and the way they structure their classroom lessons. In this paper we present data from interviews with four teachers about their teaching of differentiation and derivative. The analysis shows that there are both similarities and differences in the pedagogical approaches emphasised, and that these are related to the teachers' conceptual and representational perspectives.

Background

In order for students to learn mathematics whereby they can formulate, construct models, build understanding of concepts, learn algorithmic skills, and even prove theorems, the teacher must be able to stimulate her/his class through various forms of mathematical presentation and the associated classroom discourse (Brousseau, 1997). In this way students acquire understanding, either conceptually, with improved ability to make sense of what they are doing, and/or procedurally, with the ability to apply what they learn in solving problems, depending on the teaching emphasis, or *privileging* of approaches by their teachers (Kendal & Stacey, 2001). Therefore, while there are other factors that contribute to student understanding, we can certainly say that teacher practices in the classroom have a major influence on the enhancement of student understanding.

It seems reasonable that if a teacher possesses a limited knowledge of a concept and its related subconcepts (Chinnappan & Thomas, 1999) then they will be unlikely to provide the kind of experiences students need to construct rich conceptual thinking in the area. However, the way teachers teach, and the manner in which classroom discourse proceeds are not mediated solely by the mathematical understanding of the teachers (Cooney, 1999). They are also influenced by teachers' pedagogical content knowledge and knowledge of how student understanding can be enhanced through appropriate presentation of content, together with appropriate classroom activities (Cooney, 1999; Chinnappan and Thomas, 2001; Shulman, 1986; Simon, 1995). Thus, as the teacher ponders the design and development of lessons, s/he is guided by her/his understanding of how the teaching can best be executed, along with how the topic can best be taught. Complementing this is the aim of assisting students to construct their own understanding as they participate in the classroom discourse. In this way, these different aspects of knowledge that teachers possess influence their design of teaching-learning activities.

The meaning of mathematical entities does not arise simply from a single representation, but is distributed across a number of interacting representational systems, with each emphasising different characteristics (Lesh, 2000), and supporting one or more aspects of mathematical thinking. Thus, making multiple representations available in teaching should help to improve the capacity for learning, provided the teacher addresses the links between them (Kaput, 1998; Noss & Hoyles, 1996). Thus a key conceptual ability

which teachers can help students with is the handling of different symbolisations and representations of a given mathematical concept, and construction of the *representational versatility* to be able both to interact with the representations in a meaningful manner (Thomas & Hong, 2001), and to acquire the representational fluency (Lesh, 2000) to be able to translate between them. Hence, in view of what we have said above about the role of teacher knowledge in learning it is pertinent to consider the nature of a teacher's representational perspective on a given concept if one wants to enhance classroom situations contributing to student learning. This is especially true since there is evidence that teachers vary considerably in their representational perspective on concepts such as function (Chinnappan and Thomas, 1999, 2001; Williams, 1998). One aspect considered in this research is teachers' perspectives of derivative and differentiation (we note in passing that we may distinguish between the mathematical *object* of derivative and the mathematical *process* of differentiation) from a number of points of view, including, but not limited to, representational aspects.

Recognising the importance of the nature of teacher knowledge, this paper reports on research investigating teachers' views on differentiation and derivatives; what students should learn and how they should learn it. Our aim is to present a discussion of the teachers' content knowledge alongside their pedagogical perspectives on student learning of derivative. We will consider teachers' understanding of derivative, their views on what prerequisite knowledge students require, for derivatives and differentiation, and the pedagogical approach they employ. We will try to infer the emphases teachers make in teaching derivative, and the varying representational approaches (graphical, symbolic, or tabular) they use in helping students understand. This research is a part of a wider project that aims to document how students formulate their understanding of derivatives and differentiation when graphic or CAS calculators are integrated in the classroom. However, we recognise that without teachers' willingness to use them, and without their openness to modification of their teaching practices, potential benefits will not eventuate.

Method

In this study we have been working with four teachers from New Zealand schools, each of whom has been involved in constructing a module of work on the learning of differentiation which integrates use of GC or CAS calculators. The teachers, who were selected on the basis of their willingness to integrate the use of graphic calculators in their classroom teaching, comprise two males (A and B) and two females (C and D). Three of them are very experienced teachers while one of the males has only been teaching for two years (B). They teach in a range of schools, including two private schools (one for girls and one co-educational) and two state schools (both co-educational, but one from a high socio-economic area and one less so). The teachers were given a semi-structured interview which asked them what they think is important when teaching derivatives and differentiation, how they teach their first lessons on derivatives, and how they think students go through the process of understanding derivatives and differentiation (see sample interview questions in Figure 1). The interviews were all tape-recorded and later transcribed for analysis. Later we will be following up the interviews with classroom observation of the teaching of differentiation using the teacher-generated modules.

Teacher Interview Script
(Selected questions only)

1. The teaching of derivatives and differentiation

- Have you taught derivatives before?
- What is your first lesson on derivatives and differentiation? What is your follow up lesson?
- Could you tell me about the factors that you consider important when planning for your lessons?
- Could you give a brief outline of how you have gone about teaching derivatives?
- What related ideas do you use in introducing derivatives to your students? Why have you considered them important?
- What kind of examples do you use in teaching derivatives?

2. The learning of derivatives and differentiation

- What do you think students need to know about derivatives and differentiation?
- What key ideas should they learn?
- What skills should they master?
- What ideas do students need to know before studying derivatives?
- How do you think students learn or go through the process of mathematical understanding of derivatives and differentiation?
- What factors contribute to better learning and mathematical understanding of students?
- What emphasis in the teaching of derivatives will help the students to understand better?
- How will you measure students' success in learning derivatives and differentiation?

Figure 1. Selected questions from the interview schedule.

We aimed to gather data on the representational nature of teachers' content knowledge, and thinking about differentiation, and to relate this to their pedagogical practice.

Results and Discussion

The responses of the teachers have been interpreted and characterised according to the emphases they give, and how they create links between the graphical, symbolic, or tabular representations of derivative. They illuminate the teachers' views of their current practices, including their beliefs on what students should learn. In particular we aimed to answer the questions: how, pedadogically, do teachers approach derivative, and what is the role of representation in their teaching and what do teachers expect the students to learn about derivative and differentiation?

Prerequisite Knowledge

Clearly, for study of any area of mathematics there will be necessary prerequisite knowledge. Teacher C was emphatic in saying that she doesn't think one "can teach any maths if there is not a previous understanding of what's gone on before. That's just wasting time." When asked what they thought students should know before studying derivative, B replied "Derivatives...well you see, gradient [pause] ah...the idea of function. Ah [pause] yes [pause] I would say that they should have a fairly reasonable algebraic foundation", while A said, "They should have a reasonable algebraic background. They should have and appreciation of some basic functions. So, they should be familiar with linear function, they should be familiar with parabola."

As we can see, the teachers B and A both recognise the importance of function in the study of derivatives. According to A, students should have an appreciation of "basic functions" such as linear functions. Although, from B's experience, students in year 12 are not taught "a formal definition of function," students have encountered functions represented in the form $y=f(x)$, together with their associated graphs, becoming familiar with graphs of "lines, parabolas, cubic, hyperbolas, exponential, logs". C thinks that the idea of functions with their graphs is important because they "always use graphical approach quite so often...[Thus] their understanding of graphs and graphs of functions is fairly fundamental." A shares the same sentiment, highlighting the importance of an appreciation of the graphical representation of functions.

The remarks of A and B quoted above show that they also recognise a 'reasonable' standard of algebraic understanding to be an important prerequisite. However, A thinks that a graphical approach might be more helpful in assisting students towards an understanding: "traditionally, things have been concentrated on the algebraic method (the teaching of derivatives) and I'm quite sure that a lot of students can manipulate the algebra...but they don't have a real feel for what they are doing," and he laments that "there is not enough emphasis on graphics."

In addition, two of the teachers, A and C, consider the informal idea/concept of limit to be important and C also talks about variation and how "The whole idea of one variable changing with respect to another, that's all needs to be fairly sound and solid." While the comments made about prior learning are fairly predictable it can be noted that their comments contain a mix of skills and concepts, and cover both algebraic and graphical representations.

Derivatives: What Should Students Know?

Not surprisingly, all the teachers agree that after a course on differentiation, the basic derivatives and rules of differentiation should be known, with students able to differentiate functions from first principles as well as symbolically. However, only C and D said that the students should also be able to interpret derivative as a rate of change, thinking of graphically presented gradients as well as the algebraic dy/dx as rates of change. As D said, students should know that "differentiating is working out rate of change." It appeared that there was an emphasis on the graphical representation in the manner in which C would approach rate of change, and this may be related to her own preference.

> C: I would hope by the time they would come to looking at change, I handle most of them in a graphical point of view. So, I would hope they would have mental picture and essentially, they're looking at a rate of change at a particular moment that essentially, if they can simply grasp this is constantly changing and have that concept.

This agreed with a strong tendency apparent in the comments of three of the teachers, B and C to refer to the derivative or derived function as a 'gradient function'. This idea, which emerged from an emphasis on referring to slopes at points on graphs, was interesting because the word gradient carries with it a very strong link to the graphical representation. When asked about the gradient function, teacher B, replied that it is a "Function which tells you the gradient at a given point of a given curve." It is clear from subsequent comments though that for B this term carries over to algebraic work too, since when introducing differentiation rules he will say "...given a function, how do we obtain its gradient function? To go here and draw tangents all the time, we don't have to.... there is a short way to obtain the gradient function." This terminology is so strong that the teacher even

corrects himself when using the word derivative with reference to a graph, and makes the point that gradient function will be better understood by his students than using the phrase 'rate of change'.

> B: Well, a graph they can see. If you said, if you have a relationship of function...and then you said "sketch the graph of that function" they'll probably do that and there you start talking about the derivative of that. Well, the gradient function of that. Well, probably they'll know what you're talking about then. If you just said, find the rate of change of this with respect to that, then, they don't quite understand it as the same.

Teacher C also refers to the term gradient function for the graphical case, although she has the concepts of derivative and rate of change clearly connected to it.

> C: Overall, I want them to know that derivatives, are basically, of functions which give one the rates of change of something that is changing and that derivative which is not constant and this whole idea of gradient function because, I think, this is by far quite difficult to comprehend that you...your gradient, whereas before that, they'll be talking about gradient of straight line, but the gradient of something is constant, and then all of a sudden this gradient actually has variable in it, and it's that they find quite difficult and this whole idea of gradient function.

The question of whether some teachers see this gradient function in the graphical representation as distinct from the derivative in the algebraic representation is one which arises. Some evidence that teachers' thinking in terms of gradient functions might not be linking the two came from the interviews. The idea of derivative was mentioned by a couple of the teachers. While one of them (C) related the second derivative to "the gradient of the gradient function.", the other (B) linked it to the limit definition "That's what the actual derivative is. We're talking about...it's defined as a limit. So you know the limit of the chord, the gradient of the chord as the horizontal step approaches 0." This raises the prospect of a distinction being made in the minds of some teachers between the derivative defined algebraically as $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ (or a corresponding limit), and the gradient function as reserved for the application of a specific derivative to a graphical representation of a function. On the other hand it may be that the gradient function is the primary concept for some, such as B, who comments that "I had a set parabola to start with, $y=x^2$ and for each person, draw a tangent at a different point. And then work out the gradient of the tangent, and then tabulate it on the board and identify the relationship and the gradient function." This appears to indicate that for B the concept of gradient function is fundamental and crosses graphical, tabular and algebraic representations.

Pedagogical Approaches

When asked how they would introduce the teaching of differentiation three of the teachers spoke about using a standard graphical approach, possibly followed by procedural skills. B states that he would "usually start of talking about tangents to curves and establishing your idea of a gradient function and one or two activities. ... So getting them drawing tangents, working out gradients and in looking into the difficult gradient function and then start getting into the rules of differentiation." C describes how she tends "to teach derivative by looking at the gradient of a curve at a point, and then working through to find the gradient function.", while A says:

> I would quite often start with an activity where they would try to develop the notion of the gradient of the tangent being equivalent to the gradient of the derivative at a particular point. ... I suppose there will be some skills-based differentiation of some basic polynomials first ....I would have

introduced … a parabola and you choose two points, and you calculate the gradient between those two points. And if you get those points closer and closer together, you get the line that is closer and closer to the tangent.

However, teacher D has a quite different way of starting. She begins by trying to emphasise limits of functions before approaching graphs.

D: I normally start off by concentrating on functions, and what the function means and how functions behave in different ways; that they increase, decrease, … I generally move on to thinking about the concept of limit, what the limit means, and … I spend a lot of time concentrating on the limit not only in a symbolic way but in a verbal way… and then after that I get to talk about gradient, gradient as a rate of change…you just do the average rate of change … and make the secant smaller and smaller until it becomes the tangent.

We note that this teacher has strong links in her knowledge schemas between differentiation and a number of other related concepts, including: limit; gradient; rate of change; average rate of change; secant; and tangent. Furthermore she wants students to see these concepts in both "a symbolic way" and graphically. A agreed that the concept of limit was "a crucial idea" and also mentioned the idea of "some derivative sketching activities" early on in his teaching where, given a function, he would "ask them to …draw a graph of what the derivative would look like". He was the only teacher to raise this. Instead of referring to gradient function, teacher A described how "they need to know that derivative represents the slope of a function." In this way he keeps the derivative as a separate concept from a primary application, namely the slope or gradient of a graph.

We can say that the teachers' pedagogical approach to derivative has an emphasis on the graphical representation, relating it to tangents on curves. A even mentioned that "there is an indication of things to improve on emphasising the importance of a graphic approach to calculus and derivatives." This can be supported by D's comment when asked about the value of her use of different colours in representing the moving secant. She said that she "uses colours for the secant because it enhances the link and the connection that can be established between the average gradient and the gradient at a point. Such views may be summarised by B's comment that "visual learning is a powerful way to learn."

Though all of them have used the graphical approach of secants moving towards the tangent, D relates the symbolic representation, dy/dx, and the graphical representation of tangents to curves to rate of change. She even mentioned that topics in trigonometry, such as the tangent relation, are related to the gradient of a line and the symbol dy/dx, making all those seem to 'flow together'. Likewise, for C the links between representations are crucial, and as she describes it "[I] always want to try and be linking the graphical representation…by looking at symbolic representation…got the idea of basically go through differentiation…then when you go back and tie with the graph."

Applications and Problem Solving

From A's perspective students should be able to apply their skills to solve problems, particularly unfamiliar, non-routine ones. He explained non-routine problems to be ones where they do not apply a direct routine, where information needs to be extracted, and where students interpret what steps they need to do. D and B also see the importance of the contextual use of derivatives, with D saying that in the teaching of derivatives, the teacher should "attach stories to it in teaching." B thinks that students could understand derivatives as gradient functions but they (the students) "do not quite understand derivatives as rate of change. Thus, he "start(s) with kinematics because that is something they can

understand...they can relate to the idea of speed with the gradient." C also sees kinematics as a good route in to derivatives and a strong connection to graphs, commenting that "the whole idea of linking [a graph] to it, and is useful when you come to things like kinematics, when you talk about velocity and acceleration." Thus the teachers seem to believe that derivative as a rate of change can be understood better if taught in a familiar problem solving context.

The Role of Student Ability

One factor that the four teachers agree they consider in teaching mathematics, and in particular derivatives, is the ability range of their students. There seems to be a consensus among them that the emphasis on teaching varies depending on the calibre of the group. D describes the high-ability group as just being there and "they're like a sponge, they're sitting there waiting and whatever you throw into them, they're just so happy. Similarly for A, "if you've got high performing students, you've got to develop the concepts as clearly, and as appropriate, as possible.", and for B, "the good students can be challenged with harder problems: problems where more interpretation are required."

For the less-able students, however, B spends a lot of time "reinforcing their skills...If they can master skills, they can pass [since] skills are ones where students are able to do various things whey they don't have to decide...they do not have to think about the problem to know what is required or to know much." D describes how she will "try and keep the...class a little bit more structured so that they don't get lost...because I know that the worst thing for any student is to feel that they're lost...I do not just hurry along...[and] would just stick more to skill and more to rules...keep them comfortable." In a similar vein A admits that for weaker students he will "maybe concentrate on, perhaps, a more routine approach...getting the hang of the [pause] a lot of skill practice, differentiation, calculations of equations of tangent." This distinction between the teaching for different ability groups is interesting. Concentrating on skills rather than concepts for those perceived as of lower ability may be misplaced if a strong conceptual base is a necessary precursor to skills as preparation for good progress in learning.

Conclusion

What do we learn from these teachers? That they do not all have a common way of approaching the teaching of differentiation, and they have differing conceptions and emphases for some of the key concepts and sub-concepts. However, although the teachers involved in this research have emphasised, or *privileged* (Kendal & Stacey, 2001) particular representational ways of presenting derivative, especially the graphical one, they want their students to acquire a broad conceptual understanding of derivative, including understanding it as a gradient function, as a rate of change, as a limit, and symbolically (c.f. Thurston, 1995). In addition, in order for their students to build a fuller appreciation, they make attempts to relate derivative to contexts that the students are familiar with, thus working on their problem-solving abilities. However, these aims are constrained by a recognition of the different abilities of the students with the result that those perceived as less-able are limited to procedural methods, unable to handle challenging problems.

The privileging of the gradient function seen in two teachers' expressions would appear to be stressing a valuable object view (derivative, function) rather than simply a process view (differentiation) of the mathematics. However there is much evidence that students do not follow this but rather form a strong process view (e.g., Delos Santos & Thomas, 2001).

Furthermore, such privileging has the potential to cause cognitive obstacles to the construction of *representational fluency* (Lesh, 2000) and *representational versatility* (Thomas & Hong, 2001) which are essential for a full understanding of derivative. This could occur because, while the graphical perspective is important, it does not extend as easily as rate of change to some other important areas and applications, such as kinematics, certain maximisation problems, related rates, etc. The approach which needs to be encouraged, and which these teachers would surely endorse (C and D already clearly subscribe to it), is that suggested by Thurston's own experience (1995, p. 30), when he writes of his need to be "spending a good deal of mental time and effort digesting and practicing with each [way of conceiving of derivative], reconciling it with the others." This requires experiencing and relating conceptions of derivative across representations. It raises the question of whether the integration of technology, particularly the CAS calculator, may help teachers who are privileging a visual approach but are open to technology use, and modification of their pedagogy, to provide an environment which can improve understanding of derivatives for all students by stressing visual attributes but doing so in a manner which provides strong, dynamic links to other representations.

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